Total Number of Questions : 24

Time : 2.00 Hours

Max. Marks : 100

1. Prove : Characteristic of a field F is 0 then F is infinite. **(3 Marks)**

2. Explain : Is every field F has an infinite extension of F. **(3 Marks)**

3. Find the value of the improper integral **(3 Marks)**

   $\int_1^{\infty} \frac{1}{t^p}\,dt$ for $p \in \mathbb{R}$.

4. What is the cardinality of the set of all complex functions $f(z) = u(x, y) + i\,v(x, y)$, $z = x + iy$ such that u is a harmonic conjugate of v and v is a harmonic conjugate of u ? Justify. **(3 Marks)**

5. Find the remainder when $6^{n+2} + 7^{2n+1}$ is divided by 43. **(3 Marks)**

6. Prove : The center of the set of all $n \times n$ matrices over a field F is isomorphic to F. **(4 Marks)**

7. Find the number of similarity classes of idempotent matrices of order n over a field F. Explain your answer. **(4 Marks)**

8. Prove : S_5 is not solvable. **(4 Marks)**

9. Check whether the following sequence of functions $g_n(x) = \frac{1}{n(1+x^2)}$ converges uniformly or diverges on $\mathbb{R}$. **(4 Marks)**

10. Find the points on $\mathbb{R}^2$ where the directional derivative of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \sqrt{x^2 + y^2}$ exists. Also find the directional derivative(s) if it exists. **(4 Marks)**

11. Let f(z) be an entire function and M is a constant such that for a positive real number R and for an integer $n \geq 1$ $|f(z)| \leq M|z|^n$ for $|z| > R$. Prove that f(z) is a polynomial of degree less than or equal to n. **(4 Marks)**

12. Let X be a normed linear space, $S = \{x \in X / \|x\| \leq 1\}$ and f be a map from S into $\mathbb{R}$ such that $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in S$ and $\alpha x + \beta y \in S$ where α and β are scalars. Is there is an extension for f to all of X ? Justify. **(4 Marks)**

13. Let X, Y, Z be topological space. If $f : X \to Y$ and $g : Y \to Z$ are continuous then show that $g \circ f : X \to Z$ is continuous. **(4 Marks)**

14. A metric space X is connected iff every continuous function $f : X \to \{0, 1\}$ is not onto. **(4 Marks)**

15. Prove or disprove if A and C are connected subsets of a metric space X and if $A \subseteq B \subseteq C$ then B is connected. **(4 Marks)**

P.T.O.

16. Explain : R is 1) not algebraic
2) not finite
3) not simple
Extension of Q. **(5 Marks)**

17. The set of all algebraic numbers over Q in C is countable and algebraically closed (prove). **(5 Marks)**

18. Consider the function $f : (a, b] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x-a}$ check the continuity and uniform continuity of the function f on (a, b]. **(5 Marks)**

19. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y^2 - 1$. Find the solution of $f(x, y) = 0$. **(5 Marks)**

20. Evaluate the improper integral $\int_0^\infty \frac{\cos \alpha x}{(x^2 + \beta^2)^2} dx$, $\alpha, \beta > 0$ **(5 Marks)**

21. Let T be a bounded linear operator on a Hilbert space H. Show that there are unique self adjoint operators P and Q on H so that $T = P + iQ$. **(5 Marks)**

22. Three salesmen A, B and C visited a city on different routine. If A, B and C visit city after every 10 days, 7 days and 3 days respectively and A had last been to city 8 days ago B had last been to city Yesterday and C has visited city today. When will all three salesmen meet again ? **(5 Marks)**

23. Show that there are infinitely many primes of the form 6n + 5. **(5 Marks)**

24. A particle moves along the x-axis according to the law $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0$. If the particle is started at x = 0 with an initial velocity of 12 ft/sec to the left. Find x in term of t. **(5 Marks)**
